British Mathematical Olympiad

Round 2 : Thursday, 26 February 1998

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (2-5 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Taiwan, 13-21 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session in early July before leaving for Taiwan.

Do not turn over until **told to do so.**

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1. A booking office at a railway station sells tickets to 200 destinations. One day, tickets were issued to 3800 passengers. Show that

   (i) there are (at least) 6 destinations at which the passenger arrival numbers are the same;

   (ii) the statement in (i) becomes false if '6' is replaced by '7'.

2. A triangle ABC has $\angle BAC > \angle BCA$. A line AP is drawn so that $\angle PAC = \angle BCA$ where P is inside the triangle. A point Q outside the triangle is constructed so that PQ is parallel to AB, and BQ is parallel to AC. R is the point on BC (separated from Q by the line AP) such that $\angle PRQ = \angle BCA$.

   Prove that the circumcircle of ABC touches the circumcircle of PQR.

3. Suppose x, y, z are positive integers satisfying the equation
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{z},$$
   and let h be the highest common factor of x, y, z.

   Prove that $hxyz$ is a perfect square.

   Prove also that $h(y - x)$ is a perfect square.

4. Find a solution of the simultaneous equations
$$xy + yz + zx = 12$$
$$xyz = 2 + x + y + z$$
   in which all of x, y, z are positive, and prove that it is the only such solution.

   Show that a solution exists in which x, y, z are real and distinct.